

Order and Chaos in some Trigonometric Series: Curious Adventures of a Statistical Mechanic

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Abstract

This paper tells the story how a MAPLE-assisted quest for an interesting undergraduate problem in trigonometric series led some “amateurs” to the discovery that the one-parameter family of deterministic trigonometric series $\mathcal{S}_p : t \mapsto \sum_{n \in \mathbb{N}} \sin(n^{-p}t)$, $p > 1$, exhibits both order and apparent chaos, and how this has prompted some professionals to offer their expert insights. As to order, an elementary (undergraduate) proof is given that $\mathcal{S}_p(t) = \alpha_p \text{sign}(t)|t|^{1/p} + O(|t|^{1/(p+1)}) \forall t \in \mathbb{R}$, with explicitly computed constant α_p . As to chaos, the seemingly erratic fluctuations about this overall trend are discussed. Experts’ commentaries are reproduced as to why the fluctuations of $\mathcal{S}_p(t) - \alpha_p \text{sign}(t)|t|^{1/p}$ are presumably not Gaussian. Inspired by a central limit type theorem of Marc Kac, a well-motivated conjecture is formulated to the effect that the fluctuations of the $\lceil t^{1/(p+1)} \rceil$ -th partial sum of $\mathcal{S}_p(t)$, when properly scaled, do converge in distribution to a standard Gaussian when $t \rightarrow \infty$, though — provided that p is chosen so that the frequencies $\{n^{-p}\}_{n \in \mathbb{N}}$ are rationally linear independent; no conjecture has been forthcoming for rationally dependent $\{n^{-p}\}_{n \in \mathbb{N}}$. Moreover, following other experts’ tip-offs, the interesting relationship of the asymptotics of $\mathcal{S}_p(t)$ to properties of the Riemann ζ function is exhibited using the Mellin transform.

Key words: Riemann ζ function; Sine series; Mellin transform; Fourier transform; Tempered distributions; Deterministic chaos; Steinhaus notion of statistical independence of functions; Kac central limit theorem; Markov-Lévy method of characteristic functions.

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1 Introduction

Back in the 1990s when I was one of Jerry Percus' postdocs, I learned that Jerry's curiosity often let him explore unorthodox scientific ideas, just to see where they would lead to. In this vein, I take the invitation to celebrate the seminal contributions to statistical physics by three of its living legends: Ben and Jerry, and Michael, as a wonderful opportunity for me to follow Jerry's example and to take the three honorees (and the reader) on a curious trip into the realm of deterministic chaos without pretending that I am motivated by a physics problem — I am not! Neither do I claim any mathematical sophistication! It is just an amusing story to tell, involving several actors, interesting mathematics, a few rigorous results, and some conjectures.

The object of study is the one-parameter family of sine series

$$\mathcal{S}_p(t) = \sum_{n \in \mathbb{N}} \sin(n^{-p}t); \quad \Re p > 1, \quad (1)$$

which converges absolutely for $t \in \mathbb{R}$; it's not in [Zyg02]. Since $t^{-1}\mathcal{S}_p(t) \xrightarrow{t \rightarrow 0} \zeta(p)$, which is Riemann's Zeta function (art. VII in [Rie76]), the study of $p \mapsto \mathcal{S}_p(t)$ for fixed t , when analytically extended to $p \in \mathbb{C} \setminus \{1\}$, might be of interest to analytic number theorists. However, I don't know whether this produces anything not already known about ζ . Indeed, some relationships between $\mathcal{S}_p(t)$ and $\zeta(s)$ which go beyond the obvious one just exhibited were pointed out to me by Norm Frankel and, independently, Steve Miller, in response to my SMM 106 talk. Prompted by their insights I added section 4.2.

For the most part our attention will be on the t -dependence of $\mathcal{S}_p(t)$ for real $p > 1$. Since $\mathcal{S}_p(-t) = -\mathcal{S}_p(t)$, it suffices to discuss $\mathcal{S}_p(t)$ for positive t .

Since the sine function with the shortest wavelength contained in $\mathcal{S}_p(t)$ is $\sin(t)$, to which sine functions with ever longer wavelengths are being added, it is to be expected that $\mathcal{S}_p(t)$ is neither periodic nor quasi-periodic. Interestingly enough, the deterministic map $t \mapsto \mathcal{S}_p(t)$ exhibits apparent chaos on small scales, yet order on large ones. For example, here are two plots of $\mathcal{S}_2(t)$:

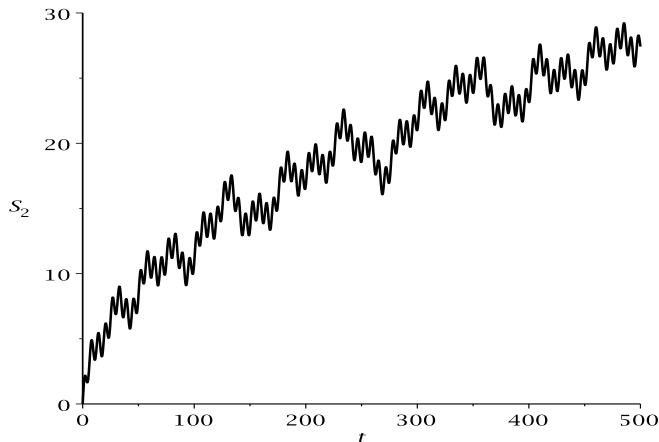


Fig.1. The 5,000-th partial sum of $\mathcal{S}_2(t)$ versus t for $0 < t < 500$.

The second one is over a 100 times larger interval of t values:

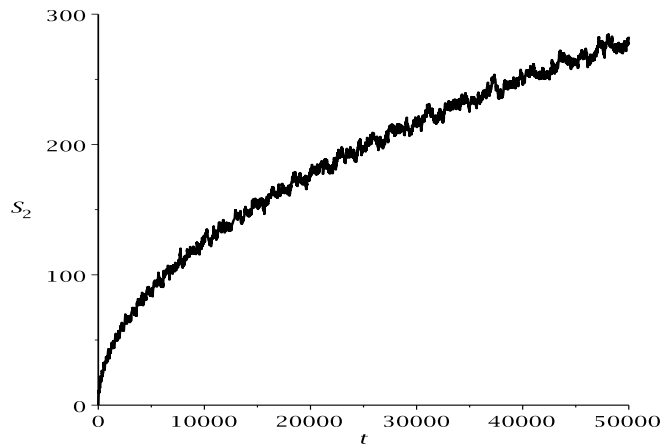


Fig.2. The 200,000-th partial sum of $S_2(t)$ versus t for $0 < t < 50,000$.

We see that relative to the range of values taken by $S_2(t)$, the seemingly erratic oscillations around their local mean appear to decrease with increasing S_2 value range, and the graph appears to converge onto a rightward opening parabola, an increase of the t domain by a factor 100 producing an increase of the S_2 value range by a factor of 10; i.e., a square root type behavior. Qualitatively similar p -th root type trends of $S_p(t)$ can be observed for other values of $p > 1$.

One of the rigorous results to be proved in this paper, with elementary means, is that $S_p(t) = \alpha_p t^{1/p} + O(t^{1/(p+1)}) \forall t > 0$, with explicitly determined α_p for all $p > 1$. This was obtained in partial collaboration with Jared Speck.

For small t , numerical evidence is given that an $O(t^{1/(p+1)})$ bound on the deviations from the overall trend is optimal, while it becomes lousy for large times. As pointed out by one of the three referees,¹ improved bounds on the deviations from the trend for large t can be obtained if the Riemann hypothesis is assumed. I summarize their comments in the added section 4.1.1.

More difficult than the determination of the trend function, but also more interesting, is the analysis of the deterministic, yet apparently chaotic fluctuations about the overall trend. A discussion of Kac's central limit theorem for sine series with rationally independent frequencies will lead us to the conjecture that the fluctuations of the $\lceil t^{1/(p+1)} \rceil$ -th partial sum of $S_p(t)$, when properly scaled, do converge in distribution to a standard Gaussian when $t \rightarrow \infty$ — provided that p is chosen so that the frequencies $\{n^{-p}\}_{n \in \mathbb{N}}$ are rationally linear independent;² no conjecture has been forthcoming for rationally dependent

¹Any similarity with the number of honorees is unintended and purely coincidental.

²It is clear that p must be chosen irrational. However, as noted by one of the referees, $p \notin \mathbb{Q}$ is not sufficient to obtain rationally linear independent frequencies of the form n^{-p} : namely, the set $\{n^{-p}\}_{n \in \mathbb{N}}$ will be rationally linear dependent whenever $p = \frac{\ln a}{\ln b}$ with integers $a > b > 1$, and this formula produces rational as well as irrational p .

$\{n^{-p}\}_{n \in \mathbb{N}}$. The stronger conjecture that $\mathcal{S}_p(t) - \alpha_p t^{1/p}$ exhibits Gaussian fluctuations, entertained by me at the time of SMM 106, is presumably wrong, as pointed out to me by two of the expert referees; see section 4.1.1. Perhaps the discussion will prompt some interested reader to work out the definitive answer using the professionals' tools.

Before we now plunge into the rigorous analysis of the functions $t \mapsto \mathcal{S}_p(t)$, I owe the reader an answer to the burning question: How come I got to dabble in the math of these sine series? After all, this is not my field of expertise! The answer is: A question by my colleague Steve Greenfield about the graph of $\mathcal{S}_2(t)$ for $0 < t < 120$ originally got me started, and the rest was curiosity about the behavior of $\mathcal{S}_2(t)$ for later t , and some fascination with what I found. So I begin with the simpler (but not so simple) behavior of $\mathcal{S}_p(t)$ at early times.

2 The early time behavior of $\mathcal{S}_p(t)$

2.1 “Can you explain the tilt?”

On April 10, 2007, Herr Dr. Prof. (emeritus) Stephen Greenfield³ sent me the following email:

“The attached picture is a graph of the 100th partial sum of the infinite series whose n th term is $\sin(x/n^2)$. You are a clever person. Why does the graph have the “tilt” that it does?”

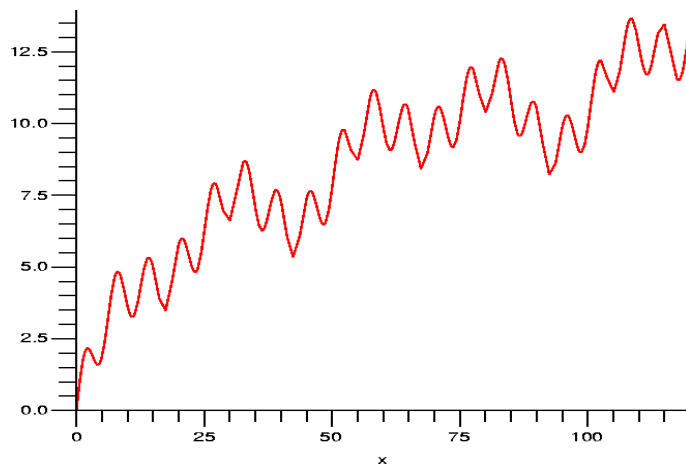


Fig.3. (Color online) Steve Greenfield’s Maple plot of $\mathcal{S}_2(x)$.

Nice question; but before I would drop everything I was doing at the time to rise to the challenge, I wanted to know why he was looking at that trigonometric series. So I went to Steve’s office three doors down the hallway to ask

³Steve likes to make fun of a German convention by calling me “Herr Dr. Prof. Kiessling;” so I assume it’s only fair when I reciprocate.

him what this was all about. As it turned out, while trying to invent some interesting unorthodox calculus problem for his honors undergraduate Maple workshop, one you won't easily find solved in a solutions manual, he had played with some unconventional trigonometric series, and this one exhibited some curious behavior: Why does the oscillating graph show some overall upward trend, instead of oscillating about zero, like more conventional sine series? Is there an explanation which a good undergraduate student could understand?

An elementary, positive lower bound which tilts upward over the full domain displayed in Steve Greenfield's picture was soon found. By itself this bound does not suffice to explain the overall shape of the graph of $\mathcal{S}_2(t)$,⁴ but at least it explained why the graph of $\mathcal{S}_2(t)$ wasn't oscillating about zero. More importantly, however, it would open the flood gates and let curiosity as to the behavior of $\mathcal{S}_2(t)$ take hold of me, and others! This bound is reproduced below.

Recall that for $t \geq 0$ we have $\sin(t/n^2) \geq t/n^2 - t^3/6n^6$. For $t/n^2 < 1$, this lower bound of $\sin(t/n^2)$ is off by 16% at worst. It can be used in the series defining $\mathcal{S}_2(t)$ whenever $n > \lceil \sqrt{t} \rceil$, where $\lceil r \rceil$ is the smallest integer not less than the real number r . Thus, writing

$$\mathcal{S}_2(t) = \sum_{n=1}^{\lceil \sqrt{t} \rceil} \sin(n^{-2}t) + \sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} \sin(n^{-2}t), \quad (2)$$

we estimate the second sum from below by

$$\sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} \sin(n^{-2}t) \geq \left(\sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} n^{-2} \right) t - \frac{1}{6} \left(\sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} n^{-6} \right) t^3. \quad (3)$$

Furthermore, by the familiar Riemann sum approximations, we estimate

$$\sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} \frac{1}{n^2} > \int_{\lceil \sqrt{t} \rceil+1}^{\infty} \frac{1}{u^2} du = \frac{1}{\lceil \sqrt{t} \rceil+1}, \quad (4)$$

$$\sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} \frac{1}{n^6} < \int_{\lceil \sqrt{t} \rceil}^{\infty} \frac{1}{u^6} du = \frac{1}{5} \frac{1}{\lceil \sqrt{t} \rceil^5}, \quad (5)$$

and so we find

$$\sum_{n=\lceil \sqrt{t} \rceil+1}^{\infty} \sin(n^{-2}t) \geq \frac{t}{\lceil \sqrt{t} \rceil+1} - \frac{1}{30} \frac{t^3}{\lceil \sqrt{t} \rceil^5}. \quad (6)$$

R.h.s.(6) is a piecewise cubic lower bound to l.h.s.(6), and therefore much easier to discuss than l.h.s.(6). In particular, it is easily seen to be positive and overall increasing roughly like \sqrt{t} ; see Fig.5 below. On the other hand, the first sum at r.h.s.(2) has just $\lceil \sqrt{t} \rceil \leq 11$ terms for the t (viz. x) interval shown in Fig.3. These were few enough to show by direct inspection that it did not have enough negative terms to overpower r.h.s.(6). In an undergraduate class

⁴I am resorting to my choice of variable " t " rather than Greenfield's " x " because the overall thrust of my paper is to think of $t \mapsto \mathcal{S}_p(t)$ as a deterministic process in time.

one would simply have to allude to the fact that a finite sum with less than a dozen terms is manageable and not go into details, though.⁵

Although I didn't plot it at the time, a Maple plot of the first sum at r.h.s.(2) shows that it itself is non-negative and tilted upward for $0 < t < 120$:

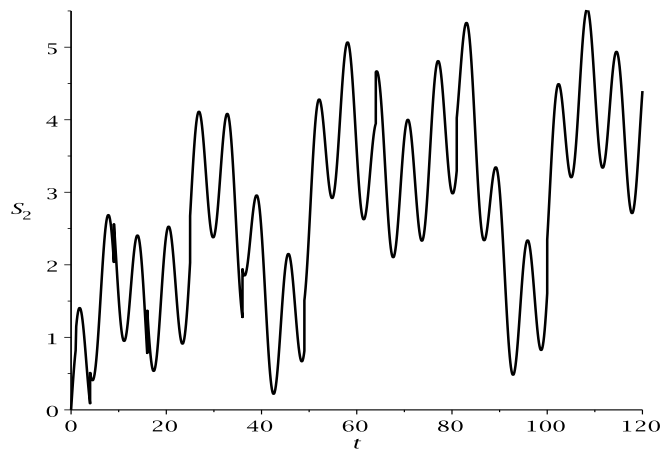


Fig.4. The first sum at r.h.s.(2).

So, curiously enough, Steve Greenfield's question applies verbatim to Fig.4! Yet, rather than trying to explain the overall upward tilt in Fig.4, one may want to try to prove merely that the first sum at r.h.s.(2) is non-negative for $0 < t < 120$. I haven't tried it, but the upshot of any such proof is: *r.h.s.(6) is an elementary lower bound to $S_2(t)$ for $0 < t < 120$* . This bound is actually quite decent; see Fig.5 below.

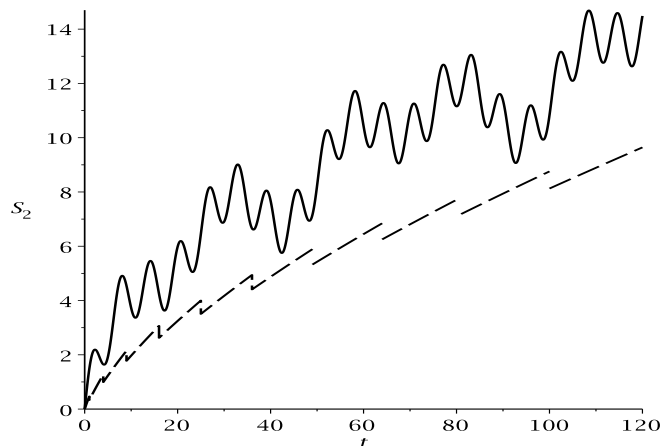


Fig.5. The 2,000th partial sum of $S_2(t)$ together with r.h.s.(6).

⁵A suitable “undergraduate bound” on $S_2(t)$ is supplied in the appendix, however.

2.2 Do all series $\mathcal{S}_p(t)$ have graphs like that of $\mathcal{S}_2(t)$?

We pause briefly to inspect the early time behavior (up to $t = 120$) of some sine series with other parameter values $p > 1$. Here are a few examples.

The first figure shows the graph of $\mathcal{S}_p(t)$ for $p = \sqrt{2}$:

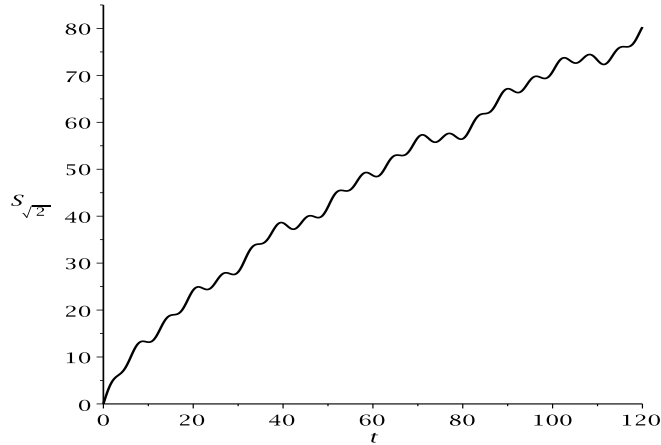


Fig.6. The 300,000-th partial sum of $\mathcal{S}_{\sqrt{2}}(t)$.

That graph looks comparable to that of $\mathcal{S}_2(t)$, only that the overall tilt is steeper, roughly by a factor six. The amplitudes of the oscillations in the graph of $\mathcal{S}_{\sqrt{2}}(t)$ appear smaller than in the graph of $\mathcal{S}_2(t)$, but appearances are misleading, for the overall range of $\mathcal{S}_{\sqrt{2}}$ values is about six times as large. In absolute terms the local fluctuations actually have increased, from a local amplitude of 1-2 in the graph of $\mathcal{S}_2(t)$ to about 2-3 in the graph of $\mathcal{S}_{\sqrt{2}}$. By the way, by “amplitude” I mean half the difference between a local maximum and its ensuing local minimum in the graph.

Next we see the graph of $\mathcal{S}_p(t)$ for $p = \sqrt{7}$. It is plotted separately because it would show merely as a “bottom dweller” if incorporated in Fig.6.

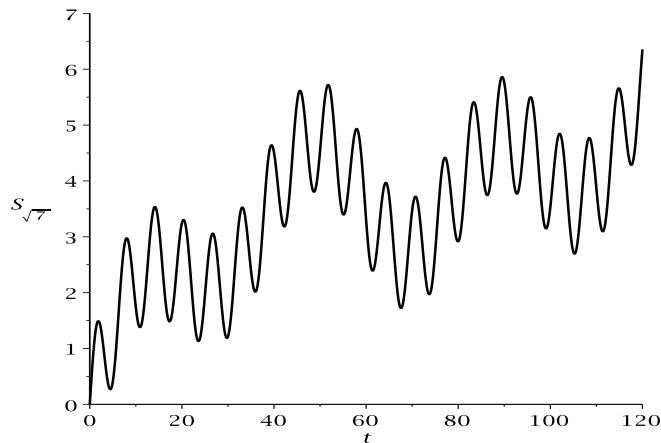


Fig.7. The 2,000-th partial sum of $\mathcal{S}_{\sqrt{7}}(t)$.

Also this graph looks comparable to that of $\mathcal{S}_2(t)$. Now the overall tilt is less steep, roughly by a factor one half. The amplitudes of the oscillations in the graph of $\mathcal{S}_{\sqrt{2}}(t)$ appear larger than in the graph of $\mathcal{S}_2(t)$, but again appearances are misleading, for the overall range of $\mathcal{S}_{\sqrt{2}}$ is about half as large. In absolute terms the local fluctuations have decreased, from a local amplitude of 1-2 in the graph of $\mathcal{S}_2(t)$ to something closer to 1 in the graph of $\mathcal{S}_{\sqrt{7}}$.

Note that the oscillations about the upward trend show $60/\pi \approx 20$ minima in all three figures, corresponding to the shortest wavelength involved.

Lest the reader now thinks that, except for the magnitude of their tilts, the graphs of $\mathcal{S}_p(t)$ look roughly alike for all values of $p > 1$, we note that

$$\lim_{p \rightarrow \infty} \mathcal{S}_p(t) = \sin(t), \quad \forall t \in \mathbb{R}. \quad (7)$$

Therefore, eventually the graph of $\mathcal{S}_p(t)$ will look essentially like that of $\sin(t)$ over the whole t interval $[0, 120]$. (I spare the reader the graph of $\sin(t)$.)

The discussion in the previous paragraph also makes it plain that the strict positivity of all displayed $\mathcal{S}_p(t)$ graphs for $t > 0$ is due to a too small sample of p values near $p = 2$. Eventually when p is large enough, the graph of $\mathcal{S}_p(t)$ will cross the t -axis. By comparing Fig.3 with Fig.7 it should come at no surprise that $p = 3$ is large enough; however, I didn't attempt to determine the critical p -value at which the first positive solution to $\mathcal{S}_p(t) = 0$ occurs, nor do I know whether this would be interesting to know.

We now turn to the more interesting problem of the overall shape of $\mathcal{S}_p(t)$.

3 The overall shape of $\mathcal{S}_p(t)$

My tending to Greenfield's question about the "tilt" of $\mathcal{S}_2(t)$ had produced the lower estimate to l.h.s.(6) given by r.h.s.(6); this estimate is bounded below by $C\sqrt{t}$, and $\asymp (29/30)\sqrt{t}$ for $t \rightarrow \infty$. Moreover, in a similar fashion one obtains an upper bound l.h.s.(6) $\leq C'\sqrt{t}$ which is asymptotic to \sqrt{t} for $t \rightarrow \infty$. These findings implied that l.h.s.(6) = $C''\sqrt{t}$ + small corrections. Furthermore, the first sum at r.h.s.(2) was bounded absolutely by $C'''\sqrt{t}$ and otherwise should be responsible for all the fluctuations visible in the plot. So when I presented Steve with my lower bound to $\mathcal{S}_2(t)$, I also told him about my conjecture that $\mathcal{S}_2(t) = \alpha_2\sqrt{t}$ + fluctuations for some constant α_2 .

The conjecture surprised Steve, for Fig.3 had suggested to him that the graph of $\mathcal{S}_2(t)$ will continue to grow on average at roughly the same rate as the overall tilt visible in Fig.3. To be fair, there isn't much of an overall concave bent of the graph of $\mathcal{S}_2(t)$ to be seen in Fig.3. Using Maple, a graph of $\mathcal{S}_2(t)$ similar to the one shown in Fig.1 was now produced, and compared with \sqrt{t} . It confirmed the $\alpha_2\sqrt{t}$ trend; however, α_2 had to be somewhat bigger than 1.

3.1 The pursuit of α_2

Enter Jared Speck, who at the time worked on his Ph.D. thesis research in relativity, advised jointly by me and my colleague Shadi Tahvildar-Zadeh, and who even may have been Greenfield's TA at the time. When I told him about Greenfield's $\mathcal{S}_2(t)$ and my conjecture about its \sqrt{t} -like trend, he didn't exactly drop whatever he was doing at the time, but the problem didn't let go of him either. By the end of April 11 (midnight, that is...) he had produced a conjecture as to what the constant α_2 could be! Jared noted that by boldly replacing the sum over $n \in \mathbb{N}$ with an integral over " dn " from 1 to ∞ , followed by the variable substitution $\mu^2 = t/n^2$, one obtains a factor \sqrt{t} that can be pulled in front of the $d\mu$ integral, and letting $t \rightarrow \infty$ in the upper limit of that $d\mu$ integral, one obtains a candidate for α_2 , namely

$$\alpha_2 = \int_0^\infty \mu^{-2} \sin \mu^2 d\mu. \quad (8)$$

April 12 was spent pondering Jared's bold proposal.

On the one hand, there was no reason to expect that⁶ $\int_1^\infty \sin(\nu^{-2}t)d\nu$ was an accurate pointwise approximation to $\mathcal{S}_2(t)$ as $t \rightarrow \infty$ because $\sin(n^{-2}t)$ hops around erratically in the interval $[-1, 1]$ when $n \mapsto n+1$ for small n (and there are more and more "small" n as t becomes large), so that one could not allude to a Riemann sum approximation. On the other hand, perhaps we could show that the difference between $\frac{1}{\sqrt{t}}\mathcal{S}_2(t)$ and $\frac{1}{\sqrt{t}}\int_1^\infty \sin(\nu^{-2}t)d\nu$ would tend to zero, so that his conjecture would be correct asymptotically.

The first impulse was to resort to the splitting (2) of the series defining $\mathcal{S}_2(t)$. The already collected evidence that the second term at r.h.s.(2) $\asymp C'''\sqrt{t}$, with $C''' \leq 1$, suggested that all that needed to be done was to prove that the first sum at r.h.s.(2) made another, though smaller, $\propto \sqrt{t}$ contribution, aside from yielding a subdominant erratic behavior. But there was an obstacle. Using the upper and lower estimates $-1 \leq \sin(\xi) \leq 1$ produces upper and lower bounds $\pm\sqrt{t}$ on the first sum at r.h.s.(2) which, while compatible with the required $\propto \sqrt{t}$ contribution, aren't good enough. There must be many near cancellations in that sum, but a term-by-term discussion, though feasible for small t , was of course out of the question for larger t .

Later that evening I realized that the key was indeed to split the sum of $\mathcal{S}_2(t)$ into two parts, but not as done in the lower estimate given in the previous section — instead, one had to split at some $n \propto \lceil t^{1/3} \rceil$ rather than at $n \propto \lceil t^{1/2} \rceil$. More precisely, with τ chosen $< \pi/2$, when t is large enough then for $n > \lceil (2t/\tau)^{1/3} \rceil$ any two consecutive arguments t/n^2 and $t/(n+1)^2$ of the sine functions would come to lie within a quarter period of sine; furthermore,

⁶To avoid unnecessary confusion, I switch to ν rather than " n " to denote the continuous integration variable, and leave n to denote the discrete summation variable.

with increasing n , for fixed t/τ , the consecutive arguments t/n^2 and $t/(n+1)^2$ would be more and more closely spaced. Put differently, for fixed sufficiently small τ , with increasing t the part of the sum of $\mathcal{S}_2(t)$ with $n > \lceil (2t/\tau)^{1/3} \rceil$ will be an increasingly better Riemann sum approximation of the integral $\int_{\lceil (2t/\tau)^{1/3} \rceil+1}^{\infty} \sin(\nu^{-2}t) d\nu$. Explicitly, if we split

$$\mathcal{S}_2(t) = \sum_{n=1}^{\lceil (2t/\tau)^{1/3} \rceil} \sin(n^{-2}t) + \sum_{n=\lceil (2t/\tau)^{1/3} \rceil+1}^{\infty} \sin(n^{-2}t), \quad (9)$$

then, for large t/τ , by Riemann sum approximation and substitution $\mu^2 = t/\nu^2$,

$$\sum_{n=\lceil (2t/\tau)^{1/3} \rceil+1}^{\infty} \sin(n^{-2}t) \approx \sqrt{t} \int_0^{t^{1/2}/\lceil (2t/\tau)^{1/3} \rceil+1} \mu^{-2} \sin \mu^2 d\mu. \quad (10)$$

For any fixed τ the upper limit of the integral at the r.h.s.(10) grows essentially $\propto t^{1/6}$, i.e. it slowly but steadily diverges to ∞ as $t \rightarrow \infty$, and so this integral converges to r.h.s.(8). Moreover, the first sum in (9) is obviously subdominant. Better yet, this erratic term should have plenty of near self-cancellations, and if one could show that it vanished on average, then even Jared's replacing of $\mathcal{S}_2(t)$ by $\int_1^{\infty} \sin(\nu^{-2}t) d\nu$ could conceivably be vindicated in an average sense. In any event, by now I had become convinced that Jared's conjecture for α_2 was right, and I sent an email to him and Steve detailing my thoughts.

An hour or so later, but still the same day (almost midnight, again), Jared replied with the following email (temporarily we are back to x instead of t):⁷

"Hello guys. Using Maple, I summed the first 200,000 terms and graphed this partial sum from $x = 0$ to $x = 50,000$. I also graphed, in yellow, $C\sqrt{x}$, where $C = \int_0^{\infty} u^{-2} \sin u^2 du$ (accurate to 8 digits). Of course, in principal, the computer could be making all sorts of round off errors, but I thought I'd take a look anyway. With all the averaging out that's going on, maybe the roundoff errors aren't significant anyway. I've attached the picture to this email. To me, the picture suggests that $C\sqrt{x}$, [with] C from above, is the right thing to try to prove. I agree with you, Michael, that a good way to proceed might be by breaking up the sum into two pieces, the 2nd of which can be approximated by the integral.

:)

~ Jared"

⁷For convenience of the reader I use L^AT_EX to display formulas which Jared described in his email. I deliberately left the amusing typo (which happens if you work until midnight!)

Here is the picture attached to his email:

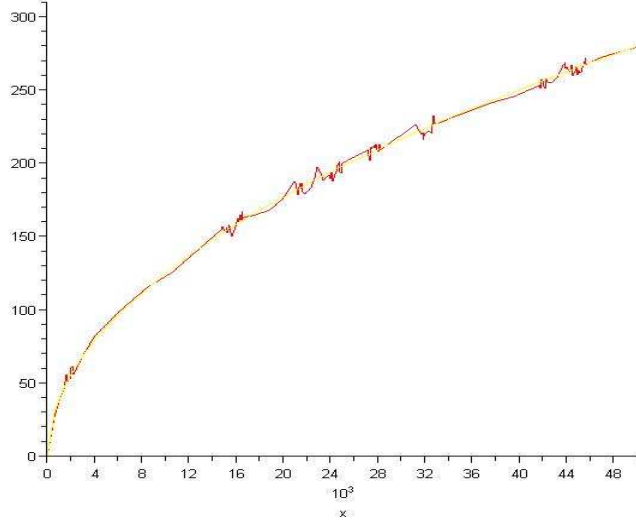


Fig.8. (Color online) Speck's 200,000-th partial sum of $S_2(x)$ together with $\alpha_2\sqrt{x}$.

Figure 8 is quite remarkable. The agreement of the displayed graph of $S_2(t)$ with that of $\alpha_2\sqrt{t}$, for α_2 given in (8), is nothing less than phenomenal. The conjecture that for *all* $t > 0$ one has $S_2(t) = \alpha_2\sqrt{t} +$ “small” fluctuations, with α_2 given in (8), had to be right!

Incidentally, note that the graph of $S_2(t)$ shown in Fig.8 displays some intriguing intermittency, as known from some turbulent flows. For most t the graph of $S_2(t)$ is barely distinguishable from that of $\alpha_2\sqrt{t}$, but every once in a while an outburst of activity is visible. How fascinating! Alas, while preparing for this presentation, when I replotted the graph with much higher resolution in Fig.2 the intriguing intermittency disappeared. Fig.8 reinforced Jared's intuition about the coefficient α_2 , which would soon be vindicated, but it was quite misleading as a guide for how to think about the fluctuations!

At noon the next day, $S_2(t)$ was the topic of the lunch conversation. In particular, Mikko Stenlund, at the time postdoc in our mathphys group, fell under the spell of the problem. Two hours later he sent me the following email:

“Hi, Michael.

For your information, if instead of the series $\sum_n \sin(x/n^2)$ one considers the corresponding integral, one gets with the aid of Fresnel integrals that the asymptotic form is $\sqrt{\pi x/2} + \sin(x/\pi^2) - \sin(x)$.

Mikko”

The conjectured α_2 integral gives an elementary value for α_2 — how wonderful! Note, though, that the indicated asymptotic expansion (replacing $x \rightarrow t$) is for $\int_1^\infty \sin(\nu^{-2}t)dt = \sqrt{t} \int_0^t \mu^{-2} \sin(\mu^2)d\mu$, not the integral at r.h.s.(10).

Hardly two hours later Jared had completed our proof of the coefficient α_2 . A little upgrading, and also the conjecture about the overall shape of $S_2(t)$ was proved. Our proof easily generalizes to all $p > 1$, to which I turn next.

3.2 The overall trend of $\mathcal{S}_p(t)$

Analogous to the reasoning for when $p = 2$, with $t > 0$, we now split the summation in the series defining $\mathcal{S}_p(t)$ at $n = \lceil (pt/\tau)^{1/(p+1)} \rceil =: N_p(t/\tau)$, thus

$$\mathcal{S}_p(t) = \sum_{n=1}^{N_p(t/\tau)} \sin(n^{-p}t) + \sum_{n=N_p(t/\tau)+1}^{\infty} \sin(n^{-p}t). \quad (11)$$

When τ is small enough (again $\tau < \pi/2$ will do when t gets large), then for $n > \lceil (pt/\tau)^{1/(p+1)} \rceil$ any two consecutive arguments t/n^p and $t/(n+1)^p$ of the sine functions will come to lie within one quarter period of sine. Moreover, with increasing n , for fixed t/τ , the consecutive arguments t/n^2 and $t/(n+1)^2$ will be more and more closely spaced. In other words, for fixed sufficiently small τ , with increasing t the part of the sum of $\mathcal{S}_p(t)$ with $n > N_p(t/\tau)$ will be an increasingly better Riemann sum approximation of the integral $\int_{N_p(t/\tau)+1}^{\infty} \sin(\nu^{-p}t) d\nu$. Thus, and after the variable substitution $\nu^{-p}t = \xi$,

$$\sum_{n=N_p(t/\tau)+1}^{\infty} \sin(n^{-p}t) \approx t^{1/p} \frac{1}{p} \int_0^{t/(N_p(t/\tau)+1)^p} \xi^{-1-1/p} \sin \xi d\xi. \quad (12)$$

Since $p > 1$, the upper limit of integration at r.h.s.(12) goes to ∞ like $At^{1/(p+1)}$ when $t \rightarrow \infty$, and the limiting integral can be evaluated by contour integration:

$$\frac{1}{p} \int_0^{\infty} \xi^{-1-1/p} \sin \xi d\xi = \Gamma\left(1 - \frac{1}{p}\right) \sin\left(\frac{\pi}{2p}\right). \quad (13)$$

Remark: Integral (13) is related by variable substitution to the *generalized Fresnel integral* $\int_0^{\infty} \sin(\eta^q) d\eta = \Gamma\left(1 + \frac{1}{q}\right) \sin\left(\frac{\pi}{2q}\right)$, which converges for $|q| > 1$.

We will sharpen “ $\approx \dots$ ” in (12) to “ $= \dots +$ a subdominant error bound.” This, a similar estimate comparing r.h.s.(12) with $t^{1/p} \times$ r.h.s.(13), and the subdominance of the first sum in (11) compared to r.h.s.(12), leads to:

Theorem 1. *For all $p > 1$, and all $t \in \mathbb{R}$, we have*

$$\mathcal{S}_p(t) = \alpha_p \operatorname{sign}(t) |t|^{1/p} + O(|t|^{1/(p+1)}), \quad (14)$$

with α_p given by r.h.s.(13).

Proof: By the anti-symmetry of $\mathcal{S}_p(t)$ it suffices to consider $t > 0$, though we need to distinguish $t \leq t_p$ and $t \geq t_p$ for some $t_p > 0$. Recall that $\lceil (pt/\tau)^{1/(p+1)} \rceil =: N_p(t/\tau)$. In all estimates below, C is a generic constant.

First of all, for $t_p > 0$ sufficiently small, we have $\mathcal{S}_p(t) = At + O(t^3)$ for $t \leq t_p$, so obviously $|\mathcal{S}_p(t) - \alpha_p t^{1/p}| \leq Ct^{1/(p+1)}$ for some C when $t \leq t_p$.

Turning to $t \geq t_p$, for the first sum at r.h.s.(11) the triangle inequality and then $|\sin \xi| \leq 1$, summing, and an obvious estimate, yield

$$\left| \sum_{n=1}^{N_p(t/\tau)} \sin(n^{-p}t) \right| \leq \lceil (pt/\tau)^{1/(p+1)} \rceil \leq Ct^{1/(p+1)}. \quad (15)$$

For the second sum at r.h.s.(11) we find (for some $\nu_n \in [n, n+1]$)

$$\left| \sum_{n=N_p(t/\tau)+1}^{\infty} \sin(n^{-p}t) - \int_{N_p(t/\tau)+1}^{\infty} \sin(\nu^{-p}t) d\nu \right| = \quad (16)$$

$$\left| \sum_{n=N_p(t/\tau)+1}^{\infty} \left(\sin(n^{-p}t) - \int_n^{n+1} \sin(\nu^{-p}t) d\nu \right) \right| = \quad (17)$$

$$\left| \sum_{n=N_p(t/\tau)+1}^{\infty} \left(\sin(n^{-p}t) - \sin(\nu_n^{-p}t) \right) \right| = \quad (18)$$

$$\left| \sum_{n=N_p(t/\tau)+1}^{\infty} \int_{t/\nu_n^p}^{t/n^p} \cos \xi d\xi \right| \leq \quad (19)$$

$$\sum_{n=N_p(t/\tau)+1}^{\infty} \int_{t/\nu_n^p}^{t/n^p} |\cos \xi| d\xi \leq \quad (20)$$

$$\sum_{n=N_p(t/\tau)+1}^{\infty} t \left(\frac{1}{n^p} - \frac{1}{\nu_n^p} \right) \leq \quad (21)$$

$$\sum_{n=N_p(t/\tau)+1}^{\infty} t \left(\frac{1}{n^p} - \frac{1}{(n+1)^p} \right) = \quad (22)$$

$$t \lceil (pt/\tau)^{1/(p+1)} + 1 \rceil^{-p} \leq Ct^{1/(p+1)}. \quad (23)$$

In this string,⁸ as for the first three equalities: (16) is manifestly true, whereas (17) holds by the mean value theorem for some $\nu_n \in [n, n+1]$, and (18) holds by the fundamental theorem of calculus; as for the ensuing three inequalities: (19) holds by the triangle inequality, (20) holds since $|\cos \xi| \leq 1$, followed by elementary integration, while (21) is due to the monotonic decrease of $\nu \mapsto \nu^{-p}$ for $p > 1$, with $\nu_n \in [n, n+1]$; the ensuing equality (22) holds because the sum at l.h.s.(22) is telescoping; lastly, inequality (23) is obvious.

For the integral in (16) the variable substitution $\nu^{-p}t = \xi$ yields

$$t^{1/p} \frac{1}{p} \int_0^{t/(N_p(t/\tau)+1)^p} \xi^{-1-1/p} \sin \xi d\xi = t^{1/p} \left[\alpha_p - \frac{1}{p} \int_{t/(N_p(t/\tau)+1)^p}^{\infty} \xi^{-1-1/p} \sin \xi d\xi \right]. \quad (24)$$

⁸I am heeding the advice Michael Fisher gave me (after reading [Kie08]) on Dec.14, 2007: “I have now had a chance to delve further into your write-up. Eventually, I found out why you say “mean field”. The answer is three totally unnumbered equations: That represents very bad practice! [...] Please do number all crucial equations in your future papers!”

Using one last time the triangle inequality and $|\sin \xi| \leq 1$, we find (for $t \geq 1$):

$$t^{1/p} \left| \int_{t/(N_p(t/\tau)+1)^p}^{\infty} \xi^{-1-1/p} \sin \xi d\xi \right| \leq \lceil (pt/\tau)^{1/(p+1)} + 1 \rceil \leq Ct^{1/(p+1)}. \quad (25)$$

The entirely elementary proof of Theorem 1 is complete. \square

Thm.1 is illustrated below by three graphs of $S_p(t)$ together with their trends $\alpha_p t^{1/p}$, for $p = 3/2$, $p = 2$, and $p = \sqrt{7}$. The t interval is always $[0, 600]$. We begin with $p = 2$ and $p = \sqrt{7}$, shown together in Fig.9.

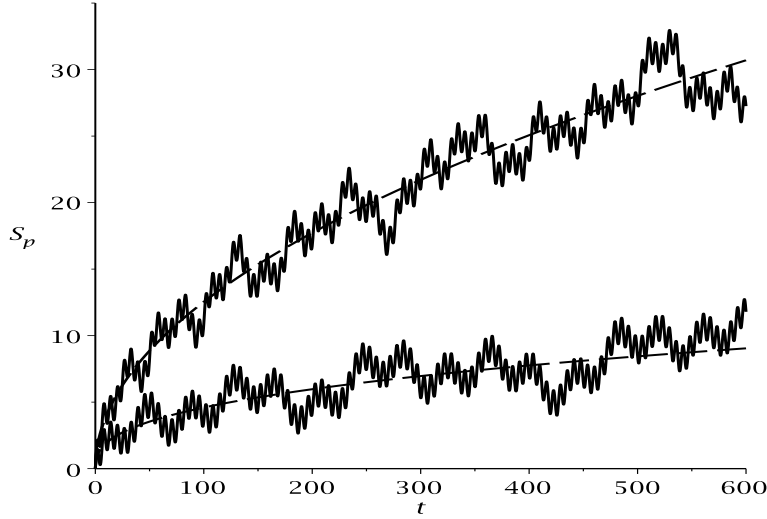


Fig.9. The 5,000-th partial sums of $S_p(t)$ for $p = 2$ and $p = \sqrt{7}$, together with their trend functions $\sqrt{\pi t/2}$ and $\Gamma(1 - \frac{1}{\sqrt{7}}) \sin(\frac{\pi}{2\sqrt{7}}) t^{1/\sqrt{7}}$, respectively.

The case $p = 3/2$, shown in Fig.10, is interesting in its own right:

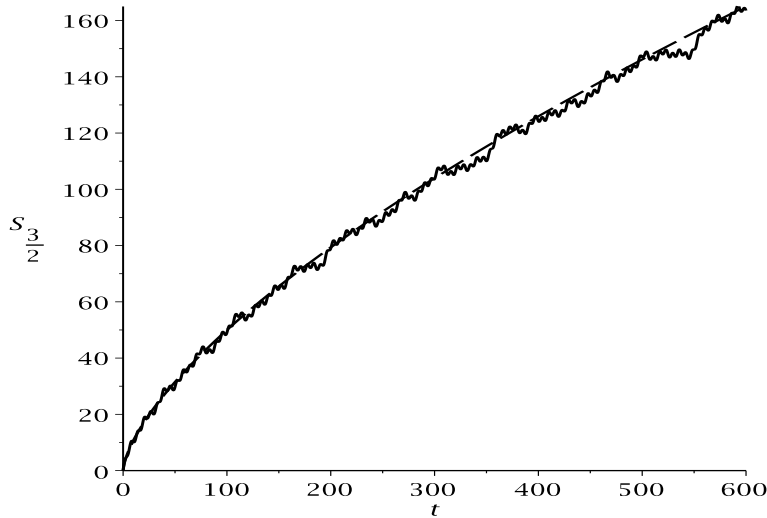


Fig.10. The 300,000-th partial sum of $S_{3/2}(t)$ together with $\Gamma(\frac{1}{3}) \sin(\frac{\pi}{3}) t^{2/3}$.

Remarkably, a “staircase” structure is clearly visible in the graph of $\mathcal{S}_{3/2}(t)$ over the t -interval $[0, 200]$, after which it gets more “noisy,” yet for $500 < t < 550$ another plateau shows. Doesn’t this call for a number-theoretical explanation?

Since for a moderately small p value like 1.5 a very large number of terms in the partial sum of $\mathcal{S}_{3/2}(t)$ was required to achieve a decently converged result, I didn’t try to push p close to 1; except, a mildly smaller, irrational $p = \sqrt{2}$ was chosen for Fig.6, with a similar expenditure in mode numbers.

In all three cases shown, the trend function $\alpha_p t^{1/p}$ truly traces the visible trend of $\mathcal{S}_p(t)$. The erratic fluctuations about the trend are more slowly growing in amplitude than the trend. Our Thm.1 says that they are bounded in amplitude by $O(t^{1/(p+1)})$. To get an idea of how accurate this bound is, I resorted to Maple to plot $\mathcal{S}_p(t) - \alpha_p t^{1/p} =: \Delta \mathcal{S}_p(t)$ together with $\pm \beta_p t^{1/(p+1)}$ for $p = 2$ and $p = \sqrt{7}$, with empirically near-optimized β_p , see Figs. 11 and 12:

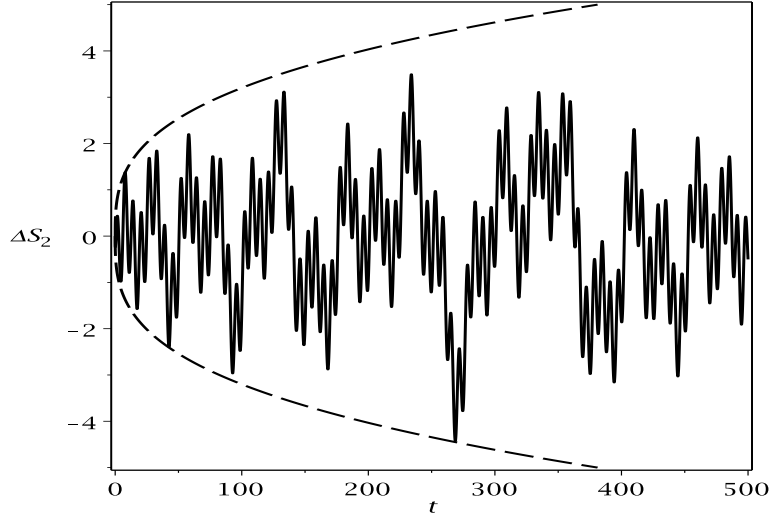


Fig.11. The 200,000-th partial sum of $\mathcal{S}_2(t) - \sqrt{\pi t/2}$ together with $\pm \frac{20}{29} t^{1/3}$.

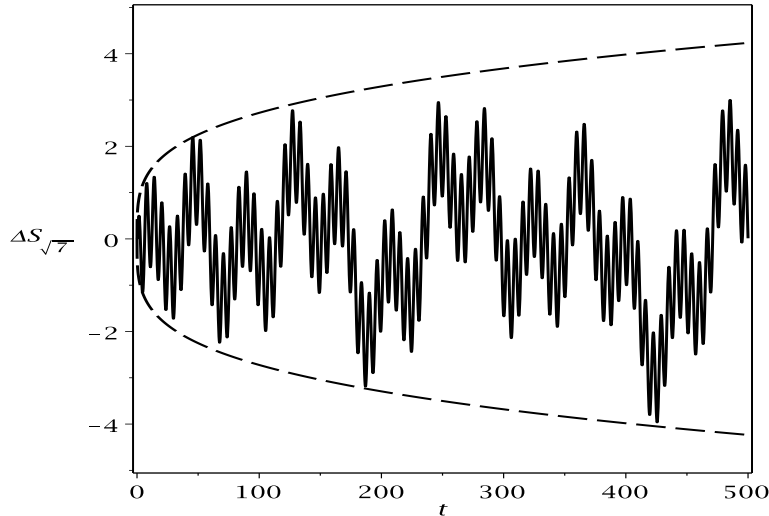


Fig.12. The 20,000-th partial sum of $\mathcal{S}_{\sqrt{7}}(t) - \Gamma(1 - \frac{1}{\sqrt{7}}) \sin(\frac{\pi}{2\sqrt{7}}) t^{1/\sqrt{7}}$ with $\pm 0.77 t^{1/(1+\sqrt{7})}$.

Figs.11 and 12 reveal that the functions $\pm\beta_p t^{1/(p+1)}$ are accurately bounding the growth of the largest fluctuation amplitudes over the shown t interval with empirically optimized β_p ; I have not tried analytically to optimize β_p . Of course, a larger sample of p values would allow a more reliable conclusion; however, since much higher precision was needed for these figures, I plotted only the cases $p = 2$ and $p = \sqrt{7}$. In section 4.2 we will use a change of variables which allows us to plot $\Delta\mathcal{S}_p$ for larger t values, see Figs. 14 & 15.

After this three-day flurry of activity the inquiry into $\mathcal{S}_p(t)$ stopped almost as abruptly as it had started, because all participants had to return to their own important businesses. However, pre-conditioned by my upbringing in statistical mechanics, I resolved to resume the inquiry into the fluctuations of $\mathcal{S}_p(t) - \alpha_p t^{1/p}$ whenever the opportunity would arise.

4 Statistics of the fluctuations of $\mathcal{S}_p(t) - \alpha_p t^{1/p}$

4.1 Kac's central limit theorem

Years later, in March 2011, while listening to Felix Izrailev's interesting presentation about quantum thermalization at Michael Kastner's STIAS workshop in Stellenbosch, South Africa, I noticed that he referred to work by Mark Kac on the central limit theorem for certain trigonometric series. I immediately wondered whether this was the information I had been waiting for to hear!

The relevant original publications are [Kac38] and [Kac43], which together with some other works by Kac were expanded into his book [Kac59]. According to the charming Kac memoir by Henry McKean (cf. p.219 in [McK90])⁹, in [Kac38] and [Kac43] the following is proved:

Theorem 2. *Let the set of frequencies $\{\omega_n\}_{n \in \mathbb{N}}$ be linearly independent over \mathbb{Q} (i.e., for any $N \in \mathbb{N}$, the only solution to $\sum_{n=1}^N z_n \omega_n = 0$ with all $z_n \in \mathbb{Z}$ is $z_1 = \dots = z_N = 0$). Let “meas” denote Lebesgue measure on \mathbb{R} . Then*

$$\lim_{N \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : a \leq \sqrt{\frac{2}{N}} \sum_{n=1}^N \sin \omega_n t \leq b \right\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}y^2} dy. \quad (26)$$

Several remarks are in order: in [Kac43], and in more detail again in [Kac59], Kac himself derives a similar formula in which cosines replace the sine functions, and with the t -average taken over the interval $[-T, T]$ rather than $[0, T]$. For Kac's cosine theorem it is obvious (since cosine is an even function) that the average over $[-T, T]$ equals the one over $[0, T]$; however, this is not true for a sum of odd sine functions. One has to go through Kac's cosines proof to see that, after replacing $[-T, T]$ with $[0, T]$ averages, one can also work with sine replacing cosine, indeed.

⁹Note a typo 1/2, instead of $1/\sqrt{2}$, in the pertinent formula on p.219 in [McK90].

At first glance Kac's Theorem looks just like "what the doctor ordered" for $\mathcal{S}_p(t)$. Unfortunately, what it says about " $\mathcal{S}_p(t)$ " (for suitable p) is *not* about $\mathcal{S}_p(t)$ — instead, it is about the infinite time averages of the family of partial sums of $\mathcal{S}_p(t)$. Of course, $\mathcal{S}_p(t)$ is *defined* as the limit $N \rightarrow \infty$ of the sequence of its N -th partial sums, but this limit does not commute with the limit $T \rightarrow \infty$ of time-averages over intervals of length T ; only for a fixed partial sum, t -averaging over $[0, T]$ and summation do commute. Kac's theorem demands that for any N -th partial sum of $\mathcal{S}_p(t)$ one first performs the limit $T \rightarrow \infty$ for the average amount of time this partial sum eventually spends in the interval $[a\sqrt{N/2}, b\sqrt{N/2}]$, then lets $N \rightarrow \infty$ (cf. [Kac43], [Kac59]).

Kac's theorem implements Steinhaus' notion of "statistical independence of functions:" the average amounts of time which individual sine functions with incommensurate frequencies spend in any infinitesimal interval within $[-1, 1]$ are eventually *i.i.d.* random variables with mean zero and standard deviation $1/\sqrt{2}$ — thus the central limit theorem type appearance of his theorem. Recall that the limit $T \rightarrow \infty$ of the unrestricted t -average over $[0, T]$ of *each* sine function vanishes whereas the t -average of its square converges to $1/2$.

By contrast, we need to take a time average *after* the infinite summation over all sine functions has been carried out and the trend function subtracted. This makes it plain that Theorem 2 above is not applicable to our problem!

Now, all this does not mean that the fluctuations of $\mathcal{S}_p(t) - \alpha_p t^{1/p}$ are not normal — they may well be ("Not so!" is the comment of one of the referees — see below). At the time of my SMM 106 presentation, under the spell of Kac's central limit theorem, I indeed conjectured that, after "suitable p -dependent rescaling," a normal law should hold for the fluctuations of $\mathcal{S}_p(t) - \alpha_p t^{1/p}$, at least for irrational p (NB: As explained in the introduction, another referee noted that $p \notin \mathbb{Q}$ won't be sufficient to guarantee the rational linear independence of the involved frequencies $\{n^{-p}\}_{n \in \mathbb{N}}$.) More precisely, a careful inspection of Kac's proof, which is based on Lévy's rigorous version of Markov's method of characteristic functions (i.e., Fourier transforms of probability measures), reveals that Markov's method should also determine the distribution of the fluctuating values of $\mathcal{S}_p(t) - \alpha_p t^{1/p}$, yet it also is clear that from some point on Kac's arguments will have to be modified. More to the point, even though our theoretical error bounds are too rough to show it, empirically the second term at r.h.s.(11) seems to be a very accurate Riemann sum approximation for the trend function when t/τ becomes moderately large, which means that $\Delta \mathcal{S}_p(t)$ should be well approximated by the first term at r.h.s.(11), which is a finite sum at each t , containing not more than $N_p(T/\tau)$ terms in the t -averages over $[0, T]$. Since $N_p(T/\tau) \asymp CT^{1/(p+1)}$, the longest wavelength in the partial sum of sines, which is averaged over $[0, T]$, grows basically $\propto T^{p/(p+1)}$, i.e. sub-linear in T so that, as T grows large, even the sine functions with the longest wavelengths in the partial sum are averaged over many cycles, infinitely many

in the limit $T \rightarrow \infty$. The upshot is the following conjecture (extending Kac's "central limit theorem"):

Conjecture 1. *Suppose p is chosen so that the frequencies $\{n^{-p}\}_{n \in \mathbb{N}}$ are rationally linear independent. Then*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \text{meas} \left\{ t \in [0, T] : a \leq \sqrt{\frac{2}{N_p(\frac{t}{\tau})}} \sum_{n=1}^{N_p(\frac{t}{\tau})} \sin(n^{-p}t) \leq b \right\} = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{1}{2}y^2} dy. \quad (27)$$

Note that Conjecture 1 is weaker than conjecturing that the fluctuations of $\mathcal{S}_p(t) - \alpha_p t^{1/p} = \Delta \mathcal{S}_p(t)$ itself are eventually normal. In any event, Conjecture 1 is interesting in its own right and, if true, may serve as an important stepping stone on the way to characterizing the fluctuations of $\Delta \mathcal{S}_p(t)$. I hope to settle this issue at some point, or that someone else will feel inspired to do so.

4.1.1 Experts' opinions on the fluctuations of $\mathcal{S}_p(t) - \alpha_p t^{1/p}$

All three referees confirmed my hunch that the questions raised above, and many more, can be answered with the techniques of analytical number theory, see the books [Vin04], [GrKo91], and the surveys [Mon94], [IwKo04]. In particular, one of the referees noted: "Assuming the Riemann hypothesis, which is quite normal in such type [of] studies, I was able to show that for any $\varepsilon > 0$,

$$\mathcal{S}_p(t) = \alpha_p t^{1/p} + t^{1/2(p+1)+\varepsilon} g_{p,\varepsilon}(\ln t), \quad (28)$$

where $g_{p,\varepsilon}$ is an L^2 -function, so that $\int_{-\infty}^{\infty} |g_{p,\varepsilon}(u)|^2 du < \infty$. I think that this gives a much stronger control of the error term than [(14)]." Another referee made a similar observation about $\mathcal{S}_p(t)$ as expressed in (28), namely that by standard techniques of Fourier analysis and analytic number theory as explained, e.g., in [GrKo91], one would be able to prove that

$$\mathcal{S}_p(t) = \alpha_p t^{1/p} + t^{1/2(p+1)} E_p(t); \quad t > 1, \quad (29)$$

where $E_p(t) = O(t^{o(1)})$ (for $t > 1$). This referee further noted that "at least in some cases (like $p = 2$) it seems possible to use the methods from the paper [HeBr92] to understand the distribution of $E_p(t)$. In any case, I think it is easy to show that for any p the distribution [of $E_p(t)$] is not going to be normal ... which would show that this conjecture [about the fluctuations of $\Delta \mathcal{S}_p(t)$] made by the author is wrong." (The referee also outlined the exponential sum obtained for $E_p(t)$ with the methods in [GrKo91], and why the distribution of $E_p(t)$ should be non-normal. Unfortunately, since I am no expert in analytical number theory, I refrain from making an (inevitably amateurish) attempt to explain these arguments here.) The third referee similarly noted that "the Gaussian behavior is violated in other examples in number theory under a similar philosophy [BCDL93] (see also [Ble92])."

I am grateful to stand rectified about my speculations about the normality of the fluctuations of $\Delta\mathcal{S}_p(t)$. I am also grateful for the observation that $p \notin \mathbb{Q}$ is insufficient to guarantee rational linear independency of the set of frequencies $\{n^{-p}\}$. As far as I can see, though, my (so revised) Conjecture 1 about the normality of fluctuations of the $\lceil t^{1/(p+1)} \rceil$ -th partial sum of $\mathcal{S}_p(t)$ is still viable.

4.2 Late- t asymptotics of $\mathcal{S}_p(t)$: Riemann's ζ function

Sometimes wondrous things happen. After reading the announcements of the SMM 106 conference talks, Norm Frankel and Steve Miller requested the pdf file of my upcoming talk. Since, as usual, I finished the preparations for my talk barely in time, they had to wait until then. Soon after, they got back to me with exciting emails, the essence of each of which I pool together. Here first is Norm Frankel (whose inimitable style I like to preserve):

“Dear Michael,

MILLE GRAZIE!

I’ve been looking forward to receiving this and will read and study it with relish [and mustard - HI]. I’ll write back when I have $> \ln(2)$ to say. ...

Using a Mellin transform, the asymptotics comes out in one line. I find $\mathcal{S}_2(x) \sim (1/4)\sqrt{2\pi/x} - \pi/4 + \text{intricate terms}$. Similarly $\mathcal{S}_p(x)$ can be readily exhibited. I seem to be differing by a factor of $1/2$

The results I sent were for $\mathcal{S}_{-2}(x)$, not $\mathcal{S}_2(x)$. OF COURSE yours is a MUCH trickier sum — back ‘gain soon. ...

I’ve just looked again at your sum. I think what I started out to do is correct in concert with analytical continuation: the Mellin transform of $\sin(n^p x) = \sin(\pi s/2)\Gamma[s]\zeta(ps)x^{-s}$. Inverting readily gives the large x asymptotics — even for $p = -2$, your series — correction terms follow.

It may be incorrect; if so, mia culps. ...

THE HAPPIEST OF HOLIDAYS FOR YOU AND YOURS!

Warmest Regards, Norm”

And here is Steve Miller:

“Dear Michael,

thank you very much for your e-mail. You really made excellent powerpoint slides—I am sure the audience appreciated them. They were very clear and entertaining. [...]¹⁰ I had not seen anything with n^{-2} like you have.

I have a few comments which I hope will be helpful to you. You asked about the connection with the Riemann ζ function at the end of your talk. If you (formally, at least) take the Mellin transform of the function $f: y \mapsto \sin(x/y^2)$ which is $Mf(s) = -\frac{1}{2}x^{s/2}\Gamma(-s/2)\sin(\pi s/4)$, then your sum is a contour integral of $Mf(s)$ times the Riemann ζ function. The Riemann ζ function has only one pole, at $s = 1$, and the residue of $Mf(s)\zeta(s)$ at $s = 1$ is your

¹⁰Steve drew my attention to the intriguing papers [ChUb07, MiSch04] on $\sin(n^2 x)$ series.

main term $\sqrt{\pi x/2}$. The only other poles of $Mf(s)\zeta(s)$ are at values of s of the form $2 + 4n$, $n \geq 0$; these give correction terms in the asymptotic expansion of Greenfield's infinite sum as $x \rightarrow 0$. So the full asymptotic expansion should come from this.

I also want to note that the functional equation of the Riemann ζ function (again, completely formally – this is essentially Poisson summation here) gives the identity that your Greenfield's sum is the sum of $g(n, x)$ over $n > 0$, where the Mellin transform of $g(z, x)$ in z is

$$-(2\pi)^{1-s} x^{(1-s)/2} \cos(\pi(s+1)/4) \Gamma(s-1) \zeta(s) / \Gamma((1-s)/2)$$

This function (without the $\zeta(s)$ factor) is a sum of hypergeometric functions

$$\begin{aligned} & \sqrt{2\pi} \sqrt{x} \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \frac{5}{4}\right\}, -\frac{1}{64} \pi^4 x^{\frac{1}{2}}\right] + \\ & \frac{2}{3} \pi^{\frac{1}{2}} x \left(-3 \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}\right\}, -\frac{1}{64} \pi^4 x^{\frac{1}{2}}\right] + \right. \\ & \left. 2 \sqrt{2\pi} \sqrt{x} \text{HypergeometricPFQ}\left[\{\}, \left\{\frac{3}{4}, \frac{5}{4}, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}\right\}, -\frac{1}{64} \pi^4 x^{\frac{1}{2}}\right] \right) \end{aligned}$$

Fig.13. The sum of hypergeometric functions Steve included as image.

Mellin inversion shows Greenfield's sum is also a sum of this function.

That dual point of view may reveal some other properties of the sums you considered in your slides. [I think I botched a calculation here, but I hope the idea and strategy was clear].

Best holiday wishes,

Steve”

I was thrilled — and immediately turned contemplative: While Norm is located half around the globe away from my office, Steve Miller's office isn't much further away from mine than Steve Greenfield's! Here I was, having the problem in the back of my mind all these years — without ever mentioning it to Steve (M.)? What if Steve G. would have sent his question to Steve M. instead? It reminded me of “Missed Opportunities” [Dys72], the beautiful Gibbs lecture by Norm Frankel's longtime friend Freeman Dyson.

Enlightened by their comments I decided to compute the late time asymptotics of $\mathcal{S}_p(t)$ beyond the leading order term. However, things aren't quite as straightforward as they seem!

First, let me flesh out what Norm and Steve wrote in their emails. Recall that the Mellin transform of a continuous function $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is

$$(\mathcal{M}f)(s) := \int_0^\infty y^{s-1} f(y) dy \quad (30)$$

wherever r.h.s.(30) is well-defined. Supposing $|f(y)| = O(y^{-a})$ for $y \downarrow 0$ and $|f(y)| = O(y^{-b})$ for $y \uparrow \infty$, with $a < b$ sharp, then $(\mathcal{M}f)(s)$ is analytic in its fundamental strip $a < \Re s < b$, where it tends to zero as $|\Im(s)| \rightarrow \infty$ (by the

Riemann–Lebesgue lemma). Writing $(\mathcal{M}f)(s) =: \tilde{f}(s)$, its inverse transform is given by the straight contour integral (in the improper Riemann sense)

$$f(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} \tilde{f}(s) ds \quad \text{any } c \in (a, b). \quad (31)$$

Turning to $\mathcal{S}_p(t)$, for $t > 0$ we define $r := t^{-1/p}$ and write $\mathcal{S}_p(t) = \mathcal{R}_p(r)$, i.e.

$$\mathcal{R}_p(r) = \sum_{n \in \mathbb{N}} \sin((nr)^{-p}); \quad p > 1. \quad (32)$$

Since $|\mathcal{R}_p(r)| = O(r^{-1})$ for $r \downarrow 0$ and $|\mathcal{R}_p(r)| = O(r^{-p})$ for $r \uparrow \infty$, for $1 < \Re s < p$ we can take the Mellin transform of $\mathcal{R}_p(r)$ to find, with obvious manipulations,

$$\tilde{\mathcal{R}}_p(s) = \int_0^\infty r^{s-1} \sum_{n \in \mathbb{N}} \sin((nr)^{-p}) dr \quad (33)$$

$$= \sum_{n \in \mathbb{N}} \int_0^\infty r^{s-1} \sin((nr)^{-p}) dr \quad (34)$$

$$= \sum_{n \in \mathbb{N}} \frac{1}{n^s} \int_0^\infty y^{s-1} \sin(y^{-p}) dy \quad (35)$$

$$= \zeta(s) \frac{1}{p} \Gamma\left(-\frac{s}{p}\right) \sin\left(-\frac{\pi s}{2p}\right). \quad (36)$$

which is analytic in $1 < \Re s < p$, its fundamental strip. And so, for $c \in (1, p)$,

$$\mathcal{R}_p(r) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} r^{-s} \zeta(s) \frac{1}{p} \Gamma\left(-\frac{s}{p}\right) \sin\left(-\frac{\pi s}{2p}\right) ds, \quad (37)$$

or, after switching back to $t = r^{-p}$, renaming s/p into ς , and introducing

$$\bar{Q}_p(\varsigma) = \zeta(p\varsigma) \Gamma(-\varsigma) \sin\left(-\frac{\pi \varsigma}{2}\right), \quad (38)$$

we find

$$\mathcal{S}_p(t) = \frac{1}{2\pi i} \int_{c/p-i\infty}^{c/p+i\infty} t^\varsigma \bar{Q}_p(\varsigma) d\varsigma. \quad (39)$$

This is how far you can get using only Euler's series for the Riemann ζ function.

Next we use that r.h.s.(36), understood as analytic extension of l.h.s.(33), is manifestly meromorphic in \mathbb{C} , having simple poles at $s = 1$ (coming from the Riemann ζ function) and at $s/p = 2n - 1$, $n \in \mathbb{N}$ (coming from the Euler Γ function); the poles of the Γ function at $s/p = 2(n - 1)$, $n \in \mathbb{N}$, are ironed out by the pertinent zeros of the sine function. Note, though, that the ζ pole and the Γ poles are located on different sides of the fundamental strip.

Therefore, if we now shift the contour in the s plane to the right, beyond all Γ poles, we obtain the Taylor series expansion of $\mathcal{S}_p(t)$ about $t = 0$, viz.

$$\mathcal{S}_p(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\zeta(p[2k+1])}{(2k+1)!} t^{2k+1}; \quad \Re p > 1. \quad (40)$$

It is readily checked that the same expansion is obtained directly from (1) by replacing $\sin(n^{-p}t)$ by its Maclaurin expansion, then exchanging the Maclaurin summation with the summation over $n \in \mathbb{N}$ given in (1), and using the Euler series of the Riemann ζ function for $s > 1$.

If, on the other hand, we shift the contour to the left just a little bit beyond the pole at $s = 1$, say to $c_p = 1 - \epsilon$, we pick up the pole's residue and obtain

$$\mathcal{S}_p(t) = \alpha_p t^{1/p} + \frac{1}{2\pi i} \int_{\frac{c_p}{p} - i\infty}^{\frac{c_p}{p} + i\infty} t^\varsigma \bar{Q}_p(\varsigma) d\varsigma, \quad (41)$$

with α_p given in (13). Incidentally, by corollary to Theorem 1, the integral at r.h.s.(41) is bounded in magnitude by $\beta_p |t|^{1/(p+1)}$.

So we see how the Mellin transform plus the residue theorem of complex analysis reproduces — in one elegant sweep — all the results we could establish with more elementary means, save the bound on the integral at r.h.s.(41). Alas, with (40) and (41) we exhaust the information about $\mathcal{S}_p(t)$ which one can extract from the poles of $\bar{Q}_p(\varsigma)$. Clearly I cannot end on such a note!

Let's see what we can learn from the fact that the fluctuations $\Delta\mathcal{S}_p(t)$ about the trend $\alpha_p t^{1/p}$ are given by the contour integral at r.h.s.(41). Proceeding now first formally, we pretend that we can shift the contour in (41) to *any* $c_p < 1$. Writing it as $\{\varsigma = c_p/p + iv', v' \in \mathbb{R}\}$ and then changing the integration variable in the contour integral at r.h.s.(41) to v' , and then to $v = v' + ic_p/p$, gives

$$\Delta\mathcal{S}_p(t) = \frac{1}{2\pi i} \int_{c_p/p - i\infty}^{c_p/p + i\infty} t^\varsigma \bar{Q}_p(\varsigma) d\varsigma \quad (42)$$

$$= t^{c_p/p} (\mathcal{F}^{-1} \bar{Q}_p(\frac{c_p}{p} + i \cdot))(\ln t) \quad (43)$$

$$= (\mathcal{F}^{-1} \bar{Q}_p(i \cdot))(\ln t), \quad (44)$$

where

$$(\mathcal{F}^{-1} f(\cdot))(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ivu} f(v) dv \quad (45)$$

denotes the formal inverse (non-unitary) Fourier transform of $f(v)$. To make proper sense out of the formal integral (45), Fourier analysis enters next.

Since the appropriate Fourier variable for $\Delta\mathcal{S}_p(t)$, $t > 0$, is $u = \ln t$, Figs.11 and 12 are a bit deceptive now. The graph of $u \mapsto \Delta\mathcal{S}_p(e^u)$, $u \in \mathbb{R}$, shown below for $p = 2$, is a better guide to one's intuition. The u interval corresponds roughly to the t interval in Fig.11, though not quite (note that $e^6 \approx 400$). Fig.14 reveals that $\Delta\mathcal{S}_2(e^u)$ has only one zero to the left of $u = 0$ and vanishes $\asymp -\alpha_2 e^{u/2}$ when $u \downarrow -\infty$. On the other side of $u = 0$, $\Delta\mathcal{S}_2(e^u)$ develops oscillations with wavelengths which become exponentially small as u gets large while their amplitudes grow with u , though bounded by the theoretical bounds $\pm\beta_2 e^{u/3}$, with empirically optimized $\beta_2 = 20/29$ (cf. Fig.11).

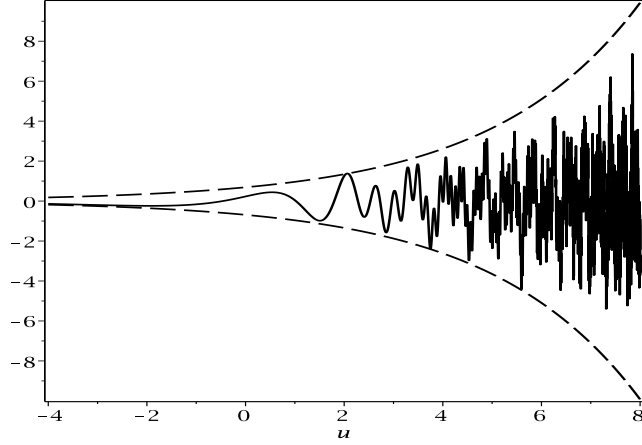


Fig.14. The graph of $u \mapsto \Delta S_2(e^u)$, $u \in (-4, 8)$, together with $\pm \frac{20}{29} \exp(u/3)$.

What Fig.14 only hints at is that beyond $u = 5.5$ the growth of the amplitudes departs more and more from our theoretical bound. This is illustrated in Fig.15, which is the continuation of Fig.14 to the right — rescaled, of course, to fit on this page.

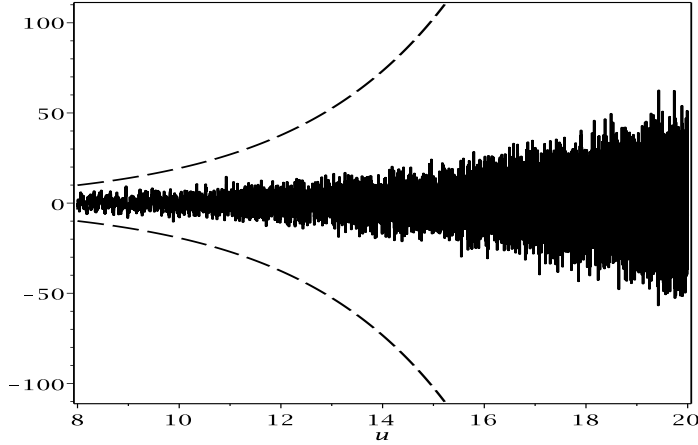


Fig.15. The graph of $u \mapsto \Delta S_2(e^u)$, $u \in (8, 20)$, together with $\pm \frac{20}{29} \exp(u/3)$.

Fig.15 leaves no doubt that the bounds $\pm \beta_2 \exp(u/3)$ become lousy for large u ; in fact, something like $\pm \gamma_2 \exp(u/5)$ traces the fluctuation amplitudes much better for $u \in (8, 20)$, but for even larger u (not shown) also this bound will outgrow the fluctuations. (NB: According to the referees, see 4.1.1, the correct bound should be $\pm \kappa_2 \exp(u/(6 + o(1)))$.) Nevertheless, as Fig.15 indicates, the fluctuations continue to grow forever. To prove their unbounded growth is not so easy, but at least it is readily shown, for all $p > 1$, that $\Delta S_p(e^u)$ does not approach 0 when $u \rightarrow \infty$ — for suppose it would, then also $\Delta S_p(t) \rightarrow 0$ when $t \uparrow \infty$, and so then does its t derivative (because $t^{(1-p)/p} \downarrow 0$ when $t \uparrow \infty$, and $S_p(t)$ contains a smallest wavelength); but the t derivative of $S_p(t)$ is a manifestly quasi-periodic function of t : a contradiction — end of proof.

The upshot of this discussion is that $u \mapsto \Delta\mathcal{S}_p(e^u)$ is a tempered distribution. Therefore, its *Fourier transform* $v \mapsto (\mathcal{F}[\Delta\mathcal{S}_p \circ \exp])(v) = \bar{Q}_p(iv)$ is to be understood *in the sense of tempered distributions* as well.

In this vein, let \mathfrak{S} denote the Schwartz space of complex \mathfrak{C}^∞ functions on \mathbb{R} which together with all their derivatives decay to zero at infinity faster than any power. If $\psi \in \mathfrak{S}$, then its Fourier transform $\mathcal{F}\psi \in \mathfrak{S}$, too, where

$$(\mathcal{F}\psi)(v) = \int_{\mathbb{R}} e^{-iuv} \psi(u) du. \quad (46)$$

The Fourier transform of a tempered distribution $g \in \mathfrak{S}'$ is then defined by

$$\int_{\mathbb{R}} (\mathcal{F}g)(v) (\mathcal{F}^{-1}\psi)(v) dv = \int_{\mathbb{R}} g(u) \psi(u) du \quad \forall \psi \in \mathfrak{S}, \quad (47)$$

where I hope to be forgiven for using the merely formal integral notation rather than a proper dual pairing notation, cf. [ReSi75].

As to the real function $u \mapsto \Delta\mathcal{S}_p(e^u)$, for our purposes it suffices to inspect its properties when integrated against the members of the family of shifted, scaled *Hermite functions* $\{\psi_n(\kappa(u-w)) \in \mathfrak{S} : w \in \mathbb{R}, \kappa \in \mathbb{R}_+\}_{n=0}^\infty$, with

$$\psi_n(u) = (2^n n! \sqrt{\pi})^{-1/2} e^{-u^2/2} H_n(u), \quad (48)$$

where

$$H_n(u) = (-1)^n e^{u^2} \frac{d^n}{du^n} e^{-u^2} \quad (49)$$

is the n -th Hermite polynomial, with $H_0 \equiv 1$. In particular, to determine the late t asymptotics of $\Delta\mathcal{S}_p(t)$, we now define quantities of the form

$$A_n^p(w; \kappa) := \int_{\mathbb{R}} \Delta\mathcal{S}_p(e^u) \psi_n(\kappa(u-w)) du \quad (50)$$

and evaluate their asymptotics as $w \rightarrow \infty$ with the help of (47).

Recalling that the Fourier transform $(\mathcal{F}[\Delta\mathcal{S}_p \circ \exp])(v) = \bar{Q}_p(iv)$, we have

$$A_n^p(w; \kappa) = \int_{\mathbb{R}} \bar{Q}_p(iv) (\mathcal{F}^{-1}[\psi_n \circ (\kappa(\cdot - w))])(v) dv. \quad (51)$$

The integrand can be recast into a more convenient format by noting that

$$(\mathcal{F}^{-1}[\psi_n \circ (\kappa(\cdot - w))])(v) = e^{i v w} \frac{1}{\kappa} (\mathcal{F}^{-1} \psi_n) \left(\frac{v}{\kappa} \right) \quad (52)$$

and by recalling that the Hermite functions are \mathfrak{L}^2 eigenfunctions for \mathcal{F} , viz.¹¹

$$(\mathcal{F}^{-1} \psi_n)(v) = \frac{i^n}{\sqrt{2\pi}} \psi_n(v). \quad (53)$$

¹¹The factor $1/\sqrt{2\pi}$ is a consequence of working with the non-unitary version of \mathcal{F} .

Note that $v \mapsto \psi_n(v)$ is even for even n and odd for odd n . Furthermore, consulting [Edw74] (or [Tit86], [Ivi03]), one sees that it follows directly from the explicit formula (38) that¹² $v \mapsto \Re(\bar{Q}_p(iv))$ is even and $v \mapsto \Im(\bar{Q}_p(iv))$ is odd, shown for $p = 2$ in Fig.16:

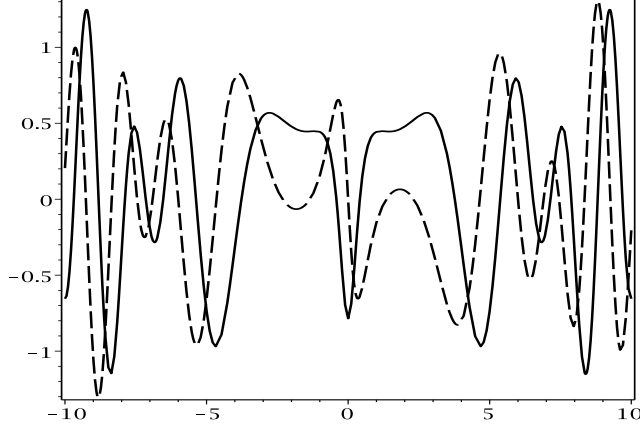


Fig.16. Real (solid) and imaginary (dashed) parts of $v \mapsto \bar{Q}_2(iv)$, $v \in \mathbb{R}$.

Therefore, the real part of the function $v \mapsto e^{ivw} \bar{Q}_p(iv)$ is even, while its imaginary part is odd. In summary, we can conclude that (51) simplifies to

$$A_n^p(w; \kappa) = (-1)^{\frac{n}{2}} \sqrt{\frac{2}{\pi}} \Re \int_0^\infty \frac{1}{\kappa} \psi_n\left(\frac{v}{\kappa}\right) \bar{Q}_p(iv) e^{ivw} dv \quad (n \text{ even}), \quad (54)$$

resp.

$$A_n^p(w; \kappa) = (-1)^{\frac{n+1}{2}} \sqrt{\frac{2}{\pi}} \Im \int_0^\infty \frac{1}{\kappa} \psi_n\left(\frac{v}{\kappa}\right) \bar{Q}_p(iv) e^{ivw} dv \quad (n \text{ odd}). \quad (55)$$

The pertinent real or imaginary part of the integrand which features in the integrals at r.h.s.(54),(55) is shown for $n = 0$ and $n = 1$ in Figs.17 & 18 below, respectively, in each case with $p = 2$, $\kappa = 1$, and $w = 60$.

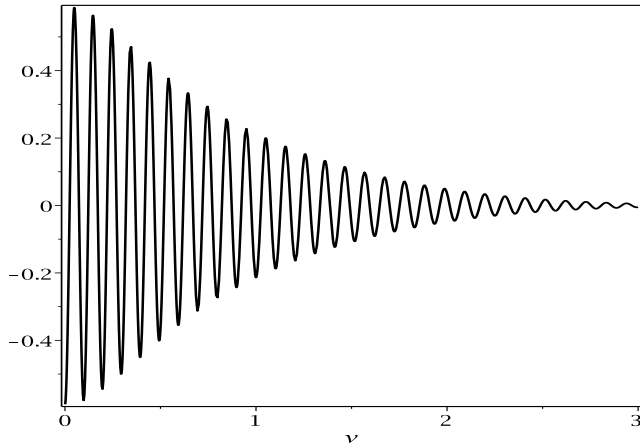


Fig.17. The graph of $\Re(e^{ivw} \bar{Q}_2(iv)) \psi_0(v)$ for $w = 60$.

¹²The Fourier transform $v \mapsto \bar{Q}_p(iv)$ of the real function $u \mapsto \Delta S_p(e^u)$ cannot be purely real or purely imaginary, for $\Delta S_p(e^u)$ is neither even nor odd, see Fig.14.

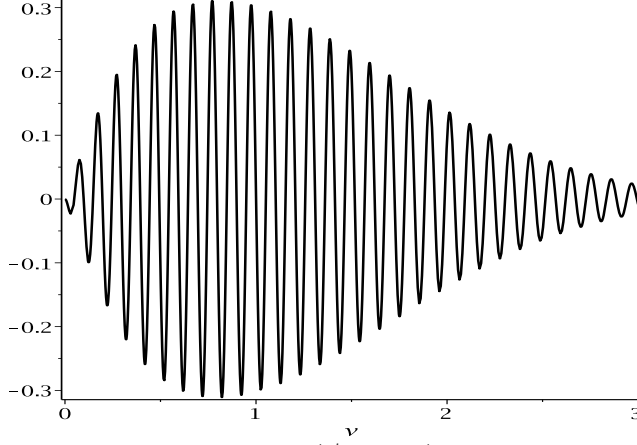


Fig.18. The graph of $\Im(e^{ivw}\bar{Q}_2(iv))\psi_1(v)$ for $w = 60$.

Since for each choice of n, p, κ the integrand of the integrals at r.h.s.(54),(55) becomes a highly oscillatory Schwartz function, by the Riemann–Lebesgue lemma the $A_n^p(w; \kappa)$ vanish in the limit $w \rightarrow \infty$, and we are interested in how they vanish asymptotically when $w \rightarrow \infty$. Since the integrand is a complex analytic function in a strip neighborhood of \mathbb{R} , their $w \rightarrow \infty$ asymptotic behavior is easily found from the full asymptotic expansion

$$\int_0^\infty \frac{1}{\kappa} \psi_n\left(\frac{v}{\kappa}\right) \bar{Q}_p(iv) e^{ivw} dv = \sum_{\ell \in \mathbb{N}} \left[\frac{2\pi}{w}\right]^\ell \frac{d^\ell}{dv^\ell} \left[\frac{1}{\kappa} \psi_n\left(\frac{v}{\kappa}\right) \bar{Q}_p(iv)\right](0) \sum_{j=0}^\ell \frac{1}{(\ell-j)!} \left[\frac{i}{2\pi}\right]^{j+1} (56)$$

(Here, $[\dots]$ does not mean “integer part.”) Inserting (56) at r.h.s.(54),(55) yields the asymptotic $w \rightarrow \infty$ expansion of $A_n^p(w; \kappa)$. For most practical matters we will only need the pertinent lowest order nonvanishing term in (56), as in the example below.

4.2.1 Application: Gaussian averages of $u \mapsto \Delta \mathcal{S}_p(e^u)$

I briefly register the bonus of the asymptotics obtained in the previous subsection: since $\psi_0(u) = \pi^{-1/4} e^{-u^2/2}$, it follows that $A_0^p(w; 1)$ is proportional to a standard Gaussian average of $\Delta \mathcal{S}_p(e^u)$ centered at w . For instance,

$$A_0^2(w; 1) = \frac{\pi^{1/4}}{\sqrt{2}} \left(\frac{1}{2}\gamma + \ln(2\pi)\right) \frac{1}{w} + O\left(\frac{1}{w^2}\right), \quad (57)$$

where $\gamma = 0.57721\dots$ is Euler’s constant. Multiplication of (57) by $\pi^{1/4}/\sqrt{2\pi}$ yields the standard Gaussian average of $\Delta \mathcal{S}_2(e^u)$ centered at w .

The road is now paved to analyze the fluctuations of $u \mapsto \Delta \mathcal{S}_p(e^u)$ as $w \rightarrow \infty$. If one thinks of the w -centered Gaussian u -average as an analogue of Kac’s uniform t -average, one can ask how much “ u -time” the function $\Delta \mathcal{S}_p(e^u)$ spends on average, centered at w , in some value interval (a, b) . Markov’s method can be applied, and one might even be able to compute the answer explicitly using the formalism of the previous subsection, whether the answer is “normal” or not. So much for the Mellin transform.

5 Epilogue

After my presentation, which ended with sect. 4.1, a young participant at the meeting came to talk to me. He was incredulous, and something close to the following conversation ensued:¹³

“Prof. Kiessling:

Why are you doing this? What does this have to do with physics?”

“Well, I am doing this because it’s fun! And didn’t I myself at the end of the talk raise the question whether there are connections to physics? Also, recall that I noted that the Schrödinger hydrogen spectrum has eigenvalues $-n^{-2}$, so n^{-2} frequencies do occur in physics, you get $\cos(n^{-2}t) + i \sin(n^{-2}t)$ as t -dependent factors in any expansion using the eigenwave functions. Of course, the series $\mathcal{S}_2(t)$ itself may not occur.”

“Exactly, how can you hope that somehow this will be useful to physics?

This is crazy!”

Truth be told, I am not sure he really said “crazy,” but he certainly gave me the impression that the thought crossed his mind. And on this anecdote I close by quoting Jerry Percus from an interview he gave a few years ago at the Courant Institute [Bal08]:

“What you want is to be a little bit crazy. You want to think of things that sound like nonsense to start with and then when you get deeper, they’re not nonsense at all.”

Acknowledgement: I thank Steve Greenfield for his original question which triggered this line of inquiry; I also wish him a happy retirement. Next I thank Jared Speck and Mikko Stenlund for their participation in nailing down α_2 . Thanks go to Norm Frankel and Steve Miller for their enlightening explanations of the relationship between $\mathcal{S}_p(t)$ and $\zeta(s)$ after my SMM 106 talk; for further enlightenment see also [NHFG92, KFGT95]. Of course, I thank all of these, and also Michael Fisher, for allowing me to quote from their emails they had sent me. I also thank Joel Lebowitz for the honorable invitation to contribute to SMM 106 and to this special issue, and the three anonymous referees for their generously offered expertise in analytical number theory (including 11 references, amongst them also [HaLi36, Fle50]) which allowed me to revise the perspective on $\mathcal{S}_p(t)$ presented in this paper; it also allowed me to correct my Conjecture 1. But most of all, I thank all three honorees for all they taught (all of) us at so many a statistical mechanics meeting, and Jerry Percus in particular — for being a wonderful mentor and friend!

¹³Many of the historical developments and anecdotes I wrote up for this talk have been faithfully reconstructed from email records, but some of it only from my memory, and so it is appropriate to quote another of Jerry’s favorites: “Don’t mistake me for the facts!”

Appendix

Here we fulfill Greenfield's request and produce lower and upper bounds on $\mathcal{S}_2(t)$ suitable for an undergraduate workshop.

First of all, pick $N \in \mathbb{N} \cup \{0\}$ and split r.h.s.(1) with $p = 2$ into two parts,

$$\mathcal{S}_2(t) = \sum_{n=1}^N \sin(n^{-2}t) + \sum_{n=N+1}^{\infty} \sin(n^{-2}t), \quad (58)$$

with $\sum_{n=1}^0 \equiv 0$. Now $\sin(\xi) \geq -1$ estimates the first sum from below by $-N$, and $\sin(\xi) \geq \xi - \xi^3/6$ for $\xi \geq 0$ is used to bound the second sum by

$$\sum_{n=N+1}^{\infty} \sin(n^{-2}t) \geq \left(\sum_{n=N+1}^{\infty} n^{-2} \right) t - \frac{1}{6} \left(\sum_{n=N+1}^{\infty} n^{-6} \right) t^3. \quad (59)$$

The converging p -series are easy to evaluate when N is not too big, using

$$\sum_{n=N+1}^{\infty} n^{-2} = \frac{\pi^2}{6} - \sum_{n=1}^N n^{-2}, \quad (60)$$

$$\sum_{n=N+1}^{\infty} n^{-6} = \frac{\pi^6}{945} - \sum_{n=1}^N n^{-6}. \quad (61)$$

And so, with the abbreviations r.h.s.(60) $\equiv A_2(N)$ and $\frac{1}{6} \times$ r.h.s.(61) $\equiv B_2(N)$,

$$\mathcal{S}_2(t) \geq -N + A_2(N)t - B_2(N)t^3 \quad \forall N \in \mathbb{N} \cup \{0\}, \quad (62)$$

and thus

$$\mathcal{S}_2(t) \geq \max\{-N + A_2(N)t - B_2(N)t^3 : 0 \leq N \leq N_*\} \quad \forall N_* \in \mathbb{N} \cup \{0\}. \quad (63)$$

It suffices to pick $N_* = 7$ to obtain a ready-to-plot small family of cubic parabolas, the pointwise maximum of which is a positive lower bound to $\mathcal{S}_2(t)$ which “tilts upward” over the whole interval $0 < t < 120$.

For the sake of completeness of the discussion I should note that with equal ease one can also produce a complementary upper bound to $\mathcal{S}_2(t)$. Namely, instead of $\sin(\xi) \geq -1$ one now uses $\sin(\xi) \leq 1$ to estimate the first sum in (58) from above by N , next one uses that $\sin(\xi) \leq \xi - \xi^3/6 + \xi^5/120$ for $\xi \geq 0$ to estimate the second sum in (58) from above by

$$\sum_{n=N+1}^{\infty} \sin(n^{-2}t) \leq \left(\sum_{n=N+1}^{\infty} n^{-2} \right) t - \frac{1}{6} \left(\sum_{n=N+1}^{\infty} n^{-6} \right) t^3 + \frac{1}{120} \left(\sum_{n=N+1}^{\infty} n^{-10} \right) t^5. \quad (64)$$

The first two converging p -series are just the same as before, the third one is new, but which is equally easy to evaluate when N is not too big, using

$$\sum_{n=N+1}^{\infty} n^{-10} = \frac{\pi^{10}}{93555} - \sum_{n=1}^N n^{-10}. \quad (65)$$

And so, with the new abbreviation $\frac{1}{120} \times \text{r.h.s.}(65) \equiv C_2(N)$, we have

$$S_2(t) \leq -N + A_2(N)t - B_2(N)t^3 + C_2(N)t^5 \quad \forall N \in \mathbb{N} \cup \{0\}, \quad (66)$$

and thus, $\forall N_* \in \mathbb{N} \cup \{0\}$,

$$S_2(t) \leq \min\{-N + A_2(N)t - B_2(N)t^3 + C_2(N)t^5 : 0 \leq N \leq N_*\}. \quad (67)$$

Once again picking $N_* = 7$ we now obtain a ready-to-plot small family of quintic polynomials, the pointwise minimum of which is a positive upper bound to $S_2(t)$ which also “tilts upward” over the whole interval $0 < t < 120$. Together with the family of lower bounds from above, this does produce an upward tilted corridor in which the graph of $S_2(t)$ must lie. This is illustrated in Fig.19.

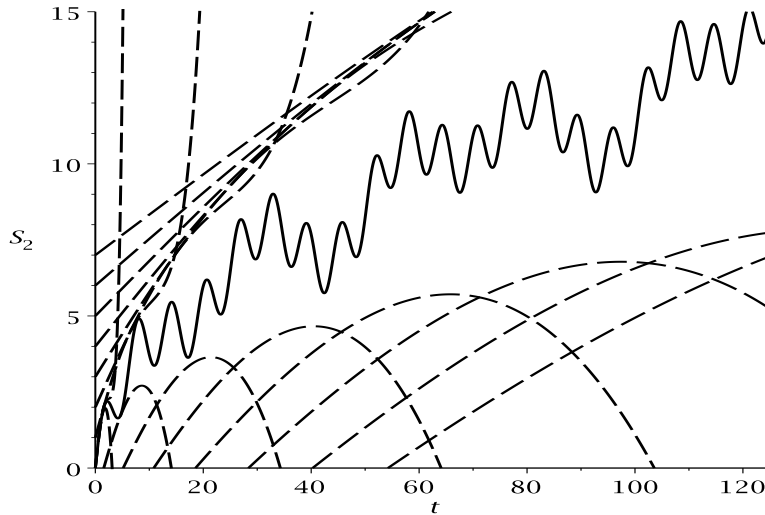


Fig.19. The 2,000-th partial sum of $S_2(t)$ together with the bounds (62) and (66) for $0 \leq N \leq 7$.

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